BASE SUBSETS OF SYMPLECTIC GRASSMANNIANS

MARK PANKOV

Abstract. Let V and V' be $2n$-dimensional vector spaces over fields F and F'. Let also $\Omega : V \times V \to F$ and $\Omega' : V' \times V' \to F'$ be non-degenerate symplectic forms. Denote by Π and Π' the associated $(2n-1)$-dimensional projective spaces. The sets of k-dimensional totally isotropic subspaces of Π and Π' will denoted by $\mathcal{G}_k$ and $\mathcal{G}'_k$, respectively. Apartments of the associated buildings intersect $\mathcal{G}_k$ and $\mathcal{G}'_k$ by so-called base subsets. We show that every mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ sending base subsets to base subsets is induced by a symplectic embedding of Π to Π'.

1. Introduction

An incidence geometry of the rank n has the following ingredients: a set $\mathcal{G}$ whose elements are called *subspaces*, a symmetric *incidence relation* on $\mathcal{G}$, and a surjective *dimension function*

$$\dim : \mathcal{G} \to \{0, 1, \dots, n-1\}$$

such that the restriction of this function to every maximal flag is bijective (flags are set of mutually incident subspaces).

A *Tits building* [13] is an incidence geometry together with a family of isomorphic subgeometries called *apartments* and satisfying a certain collection of axioms. One of these axioms says that for any two flags there is an apartment containing them.

Let us consider an incidence geometry of the rank n whose set of subspaces is denoted by $\mathcal{G}$. For every $k \in \{0, 1, \dots, n-1\}$ we denote by $\mathcal{G}_k$ the *Grassmannian* consisting of all k-dimensional subspaces. If this geometry is a building then the intersection of $\mathcal{G}_k$ with an apartment is called *the shadow* of this apartment in $\mathcal{G}_k$ [3]. In the projective and symplectic cases the intersections of apartments with Grassmannians are known as *base subsets* [8, 9, 10].

Let f be a bijective transformation of $\mathcal{G}_k$ preserving the family of the shadows of apartments. It is natural to ask: can f be extended to an automorphism of the corresponding geometry? This problem was solved in [8] for buildings of the type A_n, in this case f is induced by a collineation of the associated projective space to itself or the dual projective space (the second possibility can be realized only for the case when $n = 2k + 1$). A more general result can be found in [9].

In the present paper we show that the extension is possible for symplectic buildings.

Note that apartment preserving transformations of the chamber set (the set of maximal flags) of a spherical building are induced by automorphisms of the corresponding complex; this follows from the results given in [1].

2000 *Mathematics Subject Classification.* 51M35, 14M15.

Key words and phrases. Tits building, symplectic Grassmannians, base subsets.

2. Symplectic geometry

Let V be a $2n$-dimensional vector space over a field F, and let

$$\Omega : V \times V \to F$$

be a non-degenerate symplectic form. Denote by $\Pi = (P, \mathcal{L})$ the $(2n-1)$-dimensional projective space associated with V (points are 1-dimensional subspaces of V and lines are defined by 2-dimensional subspaces).

We say that two points $p, q \in P$ are *orthogonal* and write $p \perp q$ if

$$p =< x >,\ q =< y > \quad \text{and} \quad \Omega(x, y) = 0.$$

Similarly, two subspaces S and U of Π will be called *orthogonal* ($S \perp U$) if $p \perp q$ for any $p \in S$ and $q \in U$. The orthogonal complement to a subspace S (the maximal subspace orthogonal to S) will be denoted by $S^{\perp}$, if S is k-dimensional then the dimension of $S^{\perp}$ is equal to $2n - k - 2$.

A base $\{p_1, \dots, p_{2n}\}$ of Π is said to be *symplectic* if for each $i \in \{1, \dots, 2n\}$ there exists unique $\sigma(i) \in \{1, \dots, 2n\}$ such that

$$p_i \not\perp p_{\sigma(i)}.$$

(p_i and $p_{\sigma(i)}$ are non-orthogonal).

A subspace S of Π is called *totally isotropic* if any two points of S are orthogonal; in other words, $S \subset S^{\perp}$. The latter inclusion implies that the dimension of a totally isotropic subspace is not greater than $n - 1$.

Now consider the incidence geometry of totally isotropic subspaces. For every symplectic base B the subgeometry consisting of all totally isotropic subspaces spanned by points of B is *the symplectic apartment associated with* B. It is well-known that the incidence geometry of totally isotropic subspaces together with the family of all symplectic apartments is a building of the type C_n.

We write $\mathcal{G}_k$ for the set of all k-dimensional totally isotropic subspaces. The set of all k-dimensional totally isotropic subspaces spanned by points of a symplectic base will be called the *base subset* of $\mathcal{G}_k$ associated with (defined by) this base.

Proposition 1. *Every base subset of $\mathcal{G}_k$ consists of*

$$2^{k+1}\binom{n}{k+1}$$

elements.

Proof. Let $B = \{p_1, \dots, p_{2n}\}$ be a symplectic base and $\mathcal{B}_k$ be the associated base subset of $\mathcal{G}_k$. Denote by s_k the cardinality of $\mathcal{B}_k$.

Clearly, we can suppose that $\sigma(i) = n + i$ for every $i \le n$ (see the definition of a symplectic base). Then for each $i \in \{1, \dots, n\}$ every element of $\mathcal{B}_{n-1}$ contains precisely one of the points p_i or p_{n+i}. This implies that

$$s_{n-1} = 2^n.$$

If $k < n - 1$ then each $U \in \mathcal{B}_{k+1}$ contains $k + 2$ distinct elements of $\mathcal{B}_k$ and every $S \in \mathcal{B}_k$ is contained in $2(n - k - 1)$ distinct subspaces belonging to $\mathcal{B}_{k+1}$. Thus

$$s_k = s_{k+1}\frac{k+2}{2(n-k-1)}.$$

Step by step we get

$$s_k = s_{n-1}\frac{n}{2} \times \frac{n-1}{2\cdot 2} \times \cdots \times \frac{k+2}{2(n-k-1)}$$

which gives the claim. □

Proposition 2. *For any two k-dimensional totally isotropic subspaces there is a base subset of $\mathcal{G}_k$ containing them.*

Proposition 2 can be obtained by an immediate verification or can be drawn from the fact that for any two flags there is an apartment containing them.

3. Result

From this moment we suppose that V and V' are $2n$-dimensional vector space over fields F and F' (respectively) and

$$\Omega : V \times V \to F, \quad \Omega' : V' \times V' \to F'$$

are non-degenerate symplectic forms. Let $\Pi = (P, \mathcal{L})$ and $\Pi' = (P', \mathcal{L}')$ be the $(2n-1)$-dimensional projective spaces associated with V and V', respectively.

An injection $f : P \to P'$ is called an *embedding* of Π to Π' if it maps lines to subsets of lines and for any line $L' \in \mathcal{L}'$ there is at most one line $L \in \mathcal{L}$ such that $f(L) \subset L'$. An embedding is said to be *strong* if it sends independent subsets to independent subsets. Every strong embedding of Π to Π is induced by a semilinear injection of V to V' preserving the linear independence [4, 5, 6].

Our projective spaces have the same dimension, and strong embeddings of Π to Π' (if they exist) map bases to bases. An example given in [7] shows that strong embeddings of Π to Π' can not be characterized as mappings sending bases of Π to bases of Π. However, if $f : P \to P'$ transfers symplectic bases to symplectic bases then f is a strong embedding of Π to Π' and for any two points $p, q \in P$

$$p \perp q \implies f(p) \perp f(q) \text{ and } p \not\perp q \implies f(p) \not\perp f(q),$$

see [11]. Since a surjective embedding is a collineation, every surjection of P to P' sending symplectic bases to symplectic bases is a collineation of Π to Π' preserving the orthogonal relation.

In what follows embeddings and collineations sending symplectic bases to symplectic bases will be called *symplectic*.

Denote by $\mathcal{G}_k$ and $\mathcal{G}'_k$ the sets of k-dimensional totally isotropic subspaces of Π and Π', respectively.

Let $f : P \to P'$ be a symplectic embedding of Π to Π'. For each $S \in \mathcal{G}_k$ the subspace spanned by $f(S)$ is an element of $\mathcal{G}'_k$. The mapping

$$(f)_k : \mathcal{G}_k \to \mathcal{G}'_k$$
$$S \to \overline{f(S)}$$

(we write $\overline{X}$ for the subspace spanned by X) is an injection sending base subsets to base subsets. If f is a collineation then every $(f)_k$ is bijective. Conversely, an easy verification shows that if $(f)_k$ is bijective for certain k then f is a collineation.

Theorem 3. *If a mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ transfers base subsets to base subsets then it is induced by a symplectic embedding of Π to Π'.*

Corollary 4. *Every surjection $\mathcal{G}_k$ to $\mathcal{G}'_k$ sending base subsets to base subsets is induced by a symplectic collineation of Π to Π'.*

For $k = n-1$ these results were established in [10]. For $n = 2$ they can be drawn from well-known properties of generalized quadrangles [14].

Our proof of Theorem 3 is based on elementary properties of so-called inexact subsets (Section 4). If $k = n-1$ then all maximal inexact subsets are of the same type. The case when $k < n-1$ is more complicated: there are two different types of maximal inexact subsets.

Two elements of $\mathcal{G}_k$ are called *adjacent* if their intersection belongs to $\mathcal{G}_{k-1}$. We say that two elements of $\mathcal{G}_k$ are *ortho-adjacent* if they are orthogonal and adjacent; this is possible only if $k < n-1$. Using inexact subsets we characterize the adjacency and ortho-adjacency relations in terms of base subsets. This characterization shows that every mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ sending base subsets to base subsets is adjacency and ortho-adjacency preserving (Section 6); after that arguments in the spirit of [2] give the claim (Section 7).

4. Inexact subsets

Let $n \geq 3$ and $B = \{p_1, \dots, p_{2n}\}$ be a symplectic base of Π. Denote by $\mathcal{B}$ the base subset of $\mathcal{G}_k$ associated with B. By the definition, $\mathcal{B}$ consists of all k-dimensional subspaces

$$\overline{\{p_{i_1}, \dots, p_{i_{k+1}}\}}$$

such that

$$\{i_1, \dots, i_{k+1}\} \cap \{\sigma(i_1), \dots, \sigma(i_{k+1})\} = \emptyset.$$

If $k = m-1$ then every element of $\mathcal{B}$ contains precisely one of the points p_i or $p_{\sigma(i)}$ for each i.

We write $\mathcal{B}(+i)$ and $\mathcal{B}(-i)$ for the sets of all elements of $\mathcal{B}$ which contain p_i or do not contain p_i, respectively. For any $i_1, \dots, i_s$ and $j_1, \dots, j_u$ belonging to $\{1, \dots, 2n\}$ we define

$$\mathcal{B}(+i_1, \dots, +i_s, -j_1, \dots, -j_u) := \mathcal{B}(+i_1) \cap \dots \cap \mathcal{B}(+i_s) \cap \mathcal{B}(-j_1) \cap \dots \cap \mathcal{B}(-j_u).$$

The set of all elements of $\mathcal{B}$ incident with a subspace S will be denoted by $\mathcal{B}(S)$ (this set may be empty). Then $\mathcal{B}(-i)$ coincides with $\mathcal{B}(S)$, where S is the subspace spanned by $B \setminus \{p_i\}$. It is trivial that

$$\mathcal{B}(+i) = \mathcal{B}(+i, -\sigma(i))$$

and for the case when $k = m-1$ we have

$$\mathcal{B}(-i) = \mathcal{B}(+\sigma(i)) = \mathcal{B}(+\sigma(i), -i).$$

Let $\mathcal{R} \subset \mathcal{B}$. We say that $\mathcal{R}$ is *exact* if there is only one base subset of $\mathcal{G}_k$ containing $\mathcal{R}$; otherwise, $\mathcal{R}$ will be called *inexact*. If $\mathcal{R} \cap \mathcal{B}(+i)$ is not empty then we define $S_i(\mathcal{R})$ as the intersection of all subspaces belonging to $\mathcal{R}$ and containing p_i, and we define $S_i(\mathcal{R}) := \emptyset$ if the intersection of $\mathcal{R}$ and $\mathcal{B}(+i)$ is empty. If

$$S_i(\mathcal{R}) = p_i$$

for all i then $\mathcal{R}$ is exact; the converse fails.

Lemma 5. *Let $\mathcal{R} \subset \mathcal{B}$. Suppose that there exist i, j such that $j \neq i, \sigma(i)$ and*

$$p_j \in S_i(\mathcal{R}), \ \ p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{R}).$$

Then $\mathcal{R}$ is inexact.

Proof. On the line p_ip_j we choose a point p'_i different from p_i and p_j. The line $p_{\sigma(i)}p_{\sigma(j)}$ contains a unique point orthogonal to p'_i; we denote this point by $p'_{\sigma(j)}$. Then

$$(B \setminus \{p_i, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}$$

is a symplectic base. The associated base subset of $\mathcal{G}_k$ contains $\mathcal{R}$ and we get the claim. □

Proposition 6. *The subset $\mathcal{B}(-i)$ is inexact; moreover, if $k < n-1$ then this is a maximal inexact subset. For the case when $k = n-1$ the inexact subset $\mathcal{B}(-i)$ is not maximal.*

Proof. Let us take a point p'_i on the line $p_ip_{\sigma(i)}$ different from p_i and $p_{\sigma(i)}$. Then

$$(B \setminus \{p_i\}) \cup \{p'_i\}$$

is a symplectic base and the associated base subset of $\mathcal{G}_k$ contains $\mathcal{B}(-i)$. Hence this subset is inexact.

Let $k < n-1$. For any $j \neq i$ we can choose distinct

$$i_1, \dots, i_k \in \{1, \dots, 2n\} \setminus \{i, j, \sigma(i), \sigma(j)\}$$

such that

$$\{i_1, \dots, i_k\} \cap \{\sigma(i_1), \dots, \sigma(i_k)\} = \emptyset.$$

The subspaces spanned by

$$p_{i_1}, \dots, p_{i_k}, p_j \quad \text{and} \quad p_{\sigma(i_1)}, \dots, p_{\sigma(i_k)}, p_j$$

belong to $\mathcal{B}(-i)$. Since the intersection of these subspaces is p_j, we have

$$S_j(\mathcal{B}(-i)) = p_j \quad \text{if} \quad j \neq i. \tag{1}$$

Let U be an arbitrary taken element of

$$\mathcal{B} \setminus \mathcal{B}(-i) = \mathcal{B}(+i).$$

This subspace is spanned by p_i and some $p_{i_1}, \dots, p_{i_k}$. Since p_i is a unique point of U orthogonal to $p_{\sigma(i_1)}, \dots, p_{\sigma(i_k)}$, (1) shows that the subset

$$\mathcal{B}(-i) \cup \{U\} \tag{2}$$

is exact. This implies that the inexact subset $\mathcal{B}(-i)$ is maximal.

Now let $k = n-1$. We take an arbitrary element $U \in \mathcal{B}(+i)$. There exists j such that $p_{\sigma(j)}$ does not belongs to U. Then p_j is a point of the subspace

$$S_i(\mathcal{B}(-i) \cup \{U\}) = U.$$

Since $p_{\sigma(i)}$ belongs to every element of $\mathcal{B}(-i)$ and $p_{\sigma(j)}$ does not belongs to U,

$$S_{\sigma(j)}(\mathcal{B}(-i)) = S_{\sigma(j)}(\mathcal{B}(-i) \cup \{U\})$$

contains $p_{\sigma(i)}$. By Lemma 5, the subset (2) is inexact and the inexact subset $\mathcal{B}(-i)$ is not maximal. □

Proposition 7. *If $j \neq i, \sigma(i)$ then*

$$\mathcal{R}_{ij} := \mathcal{B}(+i, +j) \cup \mathcal{B}(+\sigma(i), +\sigma(j)) \cup \mathcal{B}(-i, -\sigma(j))$$

is a maximal inexact subset.

If $k = n-1$ then

$$\mathcal{R}_{ij} = \mathcal{B}(+i, +j) \cup \mathcal{B}(-i).$$

Proof. Since

$$S_i(\mathcal{R}_{ij}) = p_i p_j \text{ and } S_{\sigma(j)}(\mathcal{R}_{ij}) = p_{\sigma(j)} p_{\sigma(i)},$$

Lemma 5 shows that $\mathcal{R}_{ij}$ is inexact. We want to show that

$$S_l(\mathcal{R}_{ij}) = p_l \text{ if } l \neq i, \sigma(j). \tag{3}$$

Let $l \neq i, j, \sigma(i), \sigma(j)$. If $k \geq 2$ then there exists

$$i_1, \dots, i_{k-2} \in \{1, \dots, n\} \setminus \{i, j, \sigma(i), \sigma(j), l, \sigma(l)\}$$

such that

$$\{i_1, \dots, i_k\} \cap \{\sigma(i_1), \dots, \sigma(i_k)\} = \emptyset;$$

the subspaces spanned by

$$p_{i_1}, \dots, p_{i_{k-2}}, p_l, p_i, p_j \text{ and } p_{\sigma(i_1)}, \dots, p_{\sigma(i_{k-2})}, p_l, p_{\sigma(i)}, p_{\sigma(j)}$$

are elements of $\mathcal{R}_{ij}$ intersecting in the point p_l. If $k = 1$ then the lines $p_l p_{\sigma(i)}$ and $p_l p_j$ are as required.

Now we choose distinct

$$i_1, \dots, i_{k-1} \in \{1, \dots, n\} \setminus \{i, j, \sigma(i), \sigma(j)\}$$

such that

$$\{i_1, \dots, i_{k-1}\} \cap \{\sigma(i_1), \dots, \sigma(i_{k-1})\} = \emptyset$$

and consider the subspace spanned by

$$p_{i_1}, \dots, p_{i_{k-2}}, p_j, p_{\sigma(i)}.$$

This subspace intersects the subspaces spanned by

$$p_{i_1}, \dots, p_{i_{k-1}}, p_j, p_i \text{ and } p_{i_1}, \dots, p_{i_{k-1}}, p_{\sigma(i)}, p_{\sigma(j)}$$

precisely in the points p_j and $p_{\sigma(i)}$, respectively. Since all these subspaces are elements of $\mathcal{R}_{ij}$, we get (3) for $l = j, \sigma(i)$.

A direct verification shows that

$$\mathcal{B} \setminus \mathcal{R}_{ij} = \mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)).$$

Thus for every $U \in \mathcal{B} \setminus \mathcal{R}_{ij}$ one of the following possibilities is realized:

(1) $U \in \mathcal{B}(+i, -j)$ intersects $S_i(\mathcal{R}_{ij}) = p_i p_j$ by p_i,
(2) $U \in \mathcal{B}(+\sigma(j), -\sigma(i))$ intersects $S_{\sigma(j)}(\mathcal{R}_{ij}) = p_{\sigma(j)} p_{\sigma(i)}$ by $p_{\sigma(j)}$.

Since $p_{\sigma(j)}$ is a unique point of the line $p_{\sigma(j)} p_{\sigma(i)}$ orthogonal to p_i and p_i is a unique point on $p_i p_j$ orthogonal to $p_{\sigma(j)}$, the subset

$$\mathcal{R}_{ij} \cup \{U\}$$

is exact for each U belonging to $\mathcal{B} \setminus \mathcal{R}_{ij}$. Thus the inexact subset $\mathcal{R}_{ij}$ is maximal. □

The maximal inexact subsets considered in Propositions 6 and 7 will be called of the *first* and the *second* types, respectively.

Proposition 8. *Every maximal inexact subset is of the first or the second type. In particular, if $k = n - 1$ then each maximal inexact subset is of the second type.*

Proof. Let $\mathcal{R}$ be a maximal inexect subset of $\mathcal{B}$, and let B' be another symplectic base of Π such that the associated base subset of $\mathcal{G}_k$ contains $\mathcal{R}$. If certain $S_i(\mathcal{R})$ is empty then $\mathcal{R} \subset \mathcal{B}(-i)$. If $k < n-1$ then the inverse inclusion holds (since our inexact subset is maximal). If $k = n-1$ then

$$\mathcal{R} \subset \mathcal{B}(-i) \subset \mathcal{R}_{ij}$$

and $\mathcal{R} = \mathcal{R}_{ij}$.

Now suppose that each $S_i(\mathcal{R})$ is not empty. Denote by I the set of all i such that the dimension of $S_i(\mathcal{R})$ is non-zero. We take arbitrary $i \in I$ and suppose that $S_i(\mathcal{R})$ is spanned by p_i and $p_{j_1}, \dots, p_{j_u}$. If the subspaces

$$M_1 := S_{\sigma(j_1)}(\mathcal{R}), \dots, M_u := S_{\sigma(j_u)}(\mathcal{R})$$

do not contain $p_{\sigma(i)}$ then p_i belongs to $M_1^{\perp}, \dots, M_u^{\perp}$; on the other hand

$$p_{j_1} \notin M_1^{\perp}, \dots, p_{j_u} \notin M_u^{\perp}$$

and we have

$$M_1^{\perp} \cap \dots \cap M_u^{\perp} \cap S_i(\mathcal{R}) = p_i;$$

since all $S_l(\mathcal{R})$ and their orthogonal complements are spanned by points of the base B', the pointp_i belongs to B'. Therefore, there exist $i \in I$ and $j \neq i, \sigma(i)$ such that

$$p_j \in S_i(\mathcal{R}) \text{ and } p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{R}).$$

Then $\mathcal{R} = \mathcal{R}_{ij}$. □

Maximal inexact subsets of the same type have the same cardinality. These cardinalities will be denoted by $c_1(k)$ and $c_2(k)$, respectively. An immediate verification shows that each of the following possibilities

$$c_1(k) = c_2(k),\ c_1(k) < c_2(k),\ c_1(k) > c_2(k)$$

is realized.

5. Complement subsets

Let $\mathcal{B}$ be as in the previous section. We say that $\mathcal{R} \subset \mathcal{B}$ is a *complement subset* if $\mathcal{B} \setminus \mathcal{R}$ is a maximal inexact subset. A complement subset is said to be of the *first* or the *second* type if the corresponding maximal inexact subset is of the first or the second type, respectively. The compliment subsets for the maximal inexact subsets from Propositions 6 and 7 are

$$\mathcal{B}(+i) \text{ and } \mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)).$$

If $k = n-1$ then the second subset coincides with

$$\mathcal{B}(+i, +\sigma(j)) = \mathcal{B}(+i, +\sigma(j), -j, -\sigma(i)).$$

Lemma 9. *Let $k = n-1$. Then $S, U \in \mathcal{B}$ are adjacent if and only if there are precisely $\binom{k}{2}$ distinct complement subsets of $\mathcal{B}$ containing both S and U.*

Proof. Denote by m the dimension of $S \cap U$. The complement subset $\mathcal{B}(+i, +j)$ contains our subspaces if and only if p_i, p_j belong to $S \cap U$. Thus there are precisely $\binom{m+1}{2}$ distinct complement subsets of $\mathcal{B}$ containing S and U. □

Lemma 10. *Let $k < n-1$ and $\mathcal{R}$ be a complement subset of $\mathcal{B}$. If $\mathcal{R}$ is of the first type then there are precisely $4n-3$ distinct complement subsets of $\mathcal{B}$ which do not intersect $\mathcal{R}$. If $\mathcal{R}$ is of the second type then there are precisely 4 distinct complement subsets of $\mathcal{B}$ which do not intersect $\mathcal{R}$.*

To prove Lemma 10 we use the following.

Lemma 11. *Let i, i', j, j' be elements of $\{1, \dots, 2n\}$ such that $i \neq j$ and $i' \neq j'$. If the intersection of*

$$\mathcal{B}(+i, -j) \text{ and } \mathcal{B}(+i', -j')$$

is empty then one of the following possibilities is realized: $i' = \sigma(i)$, $i' = j$, $j' = i$.

Proof. Direct verification. □

Proof of Lemma 10. Let us fix $l \in \{1, \dots, 2n\}$ and consider the complement subset $\mathcal{B}(+l)$. If $\mathcal{B}(+i)$ is disjoint with $\mathcal{B}(+l)$ then $i = \sigma(l)$. If for some $i, j \in \{1, \dots, 2n\}$ the complement subset

$$\mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i))$$

does not intersect $\mathcal{B}(+l)$ then one of the following possibilities is realized:

(1) $i = \sigma(l)$, the condition $j \neq i, \sigma(i)$ shows that there are exactly $2n-2$ possibilities for j;
(2) $j = l$ and there are exactly $2n-2$ possibilities for i (since $i \neq j, \sigma(j)$).

Now fix $i, j \in \{1, \dots, 2n\}$ such that $j \neq i, \sigma(i)$ and consider the associated complement subset

$$\mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)). \tag{4}$$

There are only two complement subsets of the first type disjoint with (4):

$$\mathcal{B}(+\sigma(i)) \text{ and } \mathcal{B}(+j).$$

If

$$\mathcal{B}(+i', -j') \cup \mathcal{B}(+\sigma(j'), -\sigma(i'))$$

does not intersect (4) then one of the following two possibilities is realized:

$$i' = j,\ j' = i \text{ or } i' = \sigma(i), j' = \sigma(j)$$

(see Lemma 11). □

6. Main Lemma

Let $f : \mathcal{G}_k \to \mathcal{G}'_k$ be a mapping which sends base subsets to base subsets. Since for any two elements of $\mathcal{G}_k$ there exists a base subset containing them (Proposition 2) and the restriction of f to every base subset of $\mathcal{G}_k$ is a bijection to a base subset of $\mathcal{G}'_k$, the mapping f is injective.

In this section the following statement will be proved.

Lemma 12 (Main Lemma)**.** *Let $S, U \in \mathcal{G}_k$. Then S and U are adjacent if and only if $f(S)$ and $f(U)$ are adjacent. Moreover, for the case when $k < n-1$ the subspaces S and U are ortho-adjacent if and only if the same holds for $f(S)$ and $f(U)$.*

Let $\mathcal{B}$ be a base subset of $\mathcal{G}_k$ containing S and U. Then $\mathcal{B}' := f(\mathcal{B})$ is a base subset of $\mathcal{G}_k(\Omega')$ and the restriction $f|_{\mathcal{B}}$ is a bijection to $\mathcal{B}'$.

Lemma 13. *A subset $\mathcal{R} \subset \mathcal{B}$ is inexact if and only if $f(\mathcal{R})$ is inexact; moreover, $\mathcal{R}$ is a maximal inexact subset if and only if the same holds for $f(\mathcal{R})$.*

Proof. If $\mathcal{R}$ is inexact then there are two distinct base subsets of $\mathcal{G}_k$ containing $\mathcal{R}$ and their f-images are distinct base subsets of $\mathcal{G}'_k$ containing $f(\mathcal{R})$, hence $f(\mathcal{R})$ is inexact. The base subsets $\mathcal{B}$ and $\mathcal{B}'$ have the same number of inexact subsets and the first part of our statement is proved.

Now let $\mathcal{R}$ be a maximal inexact subset.

Suppose that $c_1(k) = c_2(k)$. Then $f(\mathcal{R})$ is an inexact subset of $\mathcal{B}'$ consisting of $c_1(k) = c_2(k)$ elements, this inexact subset is maximal. Since $\mathcal{B}$ and $\mathcal{B}'$ have the same number of maximal inexact subsets, every maximal inexact subset of $\mathcal{B}'$ is the image of a maximal inexect subset of $\mathcal{B}$.

Consider the case when $c_1(k) > c_2(k)$ (the case $c_1(k) < c_2(k)$ is similar). If $\mathcal{R}$ is of the first type then the inexect subset $f(\mathcal{R})$ consists of $c_1(k)$ elements and the inequality $c_1(k) > c_2(k)$ guarantees that this is a maximal inexact subset of the first type. The base subsets $\mathcal{B}$ and $\mathcal{B}'$ have the same number of maximal inexact subsets of the first type; thus $\mathcal{R}$ is a maximal inexact subset of the first type if and only if the same holds for $f(\mathcal{R})$. For the case when $\mathcal{R}$ is of the second type and the inexact subset $f(\mathcal{R})$ is not maximal, we take a maximal inexact subset $\mathcal{R}' \subset \mathcal{B}'$ containing $f(\mathcal{R})$; since

$$|\mathcal{R}'| > |f(\mathcal{R})| = c_2(k),$$

$\mathcal{R}'$ is of the first type and $\mathcal{R}$ is contained in the maximal inexact subset $f^{-1}(\mathcal{R}')$; the latter is impossible. Similarly, we show that every maximal inexact subset $\mathcal{R}' \subset \mathcal{B}'$ of the second type is the image of a maximal inexect subset of $\mathcal{B}$. $\square$

Lemma 14. *$\mathcal{R} \subset \mathcal{B}$ is a complement subset if and only if $f(\mathcal{R})$ is a complement subset of $\mathcal{B}'$.*

Proof. This is a simple consequence of the previous lemma. $\square$

For $k = n-1$ Main Lemma (Lemma 12) can be drawn directly from Lemmas 9 and 14. In [10] this statement was proved by a more complicated way.

Lemma 15. *The mapping $f|_{\mathcal{B}}$ together with the inverse mapping preserve types of maximal inexact and complement subsets.*

Proof. This statement is trivial if $k = n-1$ or $c_1(k) \neq c_2(k)$. For the general case this follows from Lemma 10. $\square$

We write $\mathcal{X}_i$ and $\mathcal{X}'_i$ for the sets of all i-dimensional subspaces spanned by points of the symplectic bases associated with $\mathcal{B}$ and $\mathcal{B}'$, respectively.

Lemma 16. *There exists a bijection $g : \mathcal{X}_{k+1} \to \mathcal{X}'_{k+1}$ such that*

$$f(\mathcal{B}(N)) = \mathcal{B}'(g(N))$$

for every $N \in \mathcal{X}_{k+1}$.

Proof. Lemma 15 guarantees that $f|_{\mathcal{B}}$ and the inverse mapping send maximal inexact subsets of the first type to maximal inexact subsets of the first type. This implies the existence of a bijection $h : \mathcal{X}_{2n-2} \to \mathcal{X}'_{2n-2}$ such that

$$f(\mathcal{B}(M)) = \mathcal{B}'(h(M))$$

for all $M \in \mathcal{X}_{2n-2}$. Each $N \in \mathcal{X}_{k+1}$ can be presented as the intersection of

$$M_1, \dots, M_{2n-k-2} \in \mathcal{X}_{2n-2}.$$

Then

$$g(N) := \bigcap_{i=1}^{2n-k-2} h(M_i)$$

is as required. □

Now we prove Lemma 12 for $k < n-1$. Two subspaces $S, U \in \mathcal{B}$ are adjacent if and only if they belong to $\mathcal{B}(T)$ for certain $T \in \mathcal{X}_{k+1}$; moreover, S and U are ortho-adjacent if and only if $\mathcal{B}(T)$ consists of $k+2$ elements (in other words, T is totally isotropic). The required statement follows from Lemma 16.

7. Proof of Theorem 3

Let M, N be a pair of incident subspaces of Π such that $\dim M < k < \dim N$. We put $[M, N]_k$ for the set of k-dimensional subspaces of Π incident with both M and N; for the case when $M = \emptyset$ or $N = P$ we write $(N]_k$ or $[M)_k$, respectively.

We say that $\mathcal{X} \subset \mathcal{G}_k$ is an A-*subset* if any two distinct elements of $\mathcal{X}$ are adjacent.

Example 17. If $k < n-1$ and N is an element of $\mathcal{G}_{k+1}$ then $(N]_k$ is a maximal A-subset of $\mathcal{G}_k$. Subsets of such type will be called *tops*. Any two distinct elements of a top are ortho-adjacent.

Example 18. If M belongs to $\mathcal{G}_{k-1}$ then

$$[M, M^{\perp}]_k = [M)_k \cap \mathcal{G}_k$$

is a maximal A-subset of $\mathcal{G}_k$. Such maximal A-subsets are known as *stars*, they contain non-orthogonal elements.

Fact 19 ([2],[12])**.** *Each A-subset is contained in a maximal A-subset. Every maximal A-subset of $\mathcal{G}_{n-1}$ is a star. If $k < n-1$ then every maximal A-subset of $\mathcal{G}_k$ is a top or a star.*

The first part of Lemma 12 says that f transfers A-subsets to A-subsets. The second part of Lemma 12 guarantees that stars go to subsets of stars. In other words, for any $M \in \mathcal{G}_{k-1}$ there exists $M' \in \mathcal{G}'_{k-1}$ such that

$$f([M, M^{\perp}]_k) \subset [M', M'^{\perp}]_k. \tag{5}$$

Suppose that

$$f([M, M^{\perp}]_k) \subset [M'', M''^{\perp}]_k$$

for other $M'' \in \mathcal{G}'_{k-1}$. Then $f([M, M^{\perp}]_k)$ is contained in the intersection of $[M', M'^{\perp}]_k$ and $[M'', M''^{\perp}]_k$. This intersection is not empty only if $M' = M''$ or M' and M'' are ortho-adjacent; but in the second case our intersection consists of one element. Thus there is unique $M' \in \mathcal{G}'_{k-1}$ satisfying (5). We have established the existence of a mapping

$$g : \mathcal{G}_{k-1} \to \mathcal{G}'_{k-1}$$

such that

$$f([M, M^{\perp}]_k) \subset [g(M), g(M)^{\perp}]_k$$

for every $M \in \mathcal{G}_{k-1}$. It is easy to see that

$$g((N]_{k-1}) \subset (f(N)]_{k-1} \quad \forall\, N \in \mathcal{G}_k. \tag{6}$$

Now we show that g *sends base subsets to base subsets*.

Proof. Let $\mathcal{B}_{k-1}$ be a base subset of $\mathcal{G}_{k-1}$ and B be the associated symplectic base. This base defines a base subset $\mathcal{B} \subset \mathcal{G}_k$. Now let B' be the symplectic base associated with the base subset $\mathcal{B}' := f(\mathcal{B})$ and $\mathcal{B}'_{k-1}$ be the base subset of $\mathcal{G}'_{k-1}$ defined by B'. If $S \in \mathcal{B}_{k-1}$ then we take $U_1, U_2 \in \mathcal{B}$ such that $S = U_1 \cap U_2$, and (6) shows that

$$g(S) = f(U_1) \cap f(U_2) \in \mathcal{B}_{k-1}.$$

Thus $g(\mathcal{B}_{k-1})$ is contained in $\mathcal{B}'_{k-1}$. Suppose that $g(\mathcal{B}_{k-1})$ is a proper subset of $\mathcal{B}'_{k-1}$. Then $g(S) = g(U)$ for some distinct $S, U \in \mathcal{B}_{k-1}$. The f-image of

$$\mathcal{B}(S) = \mathcal{B} \cap [S, S^{\perp}]_k$$

is contained in

$$\mathcal{B}'(g(S)) = \mathcal{B}' \cap [g(S), g(S)^{\perp}]_k.$$

Since these sets have the same cardinality,

$$f(\mathcal{B}(S)) = \mathcal{B}'(g(S)).$$

Similarly,

$$f(\mathcal{B}(U)) = \mathcal{B}'(g(U)).$$

The equality $f(\mathcal{B}(S)) = f(\mathcal{B}(U))$ contradicts the injectivity of f. Hence $g(\mathcal{B}_{k-1})$ coincides with $\mathcal{B}'_{k-1}$. □

If $k = 1$ then the mapping $g : P \to P'$ sends symplectic bases to symplectic bases, by [11] g is a symplectic embedding of Π to Π', and we have $f = (g)_1$.

Now suppose that $k > 1$ and g is induced by a symplectic embedding h of Π to Π'. Let us consider an arbitrary element $S \in \mathcal{G}_k$ and take ortho-adjacent $M, N \in \mathcal{G}_{k-1}$ such that $S = \overline{M \cup N}$. Then

$$\{S\} = [M, M^{\perp}]_k \cap [N, N^{\perp}]_k$$

and $f(S)$ belongs to the intersection of $[g(M), g(M)^{\perp}]_k$ and $[g(N), g(N)^{\perp}]_k$. Since

$$g(M) = \overline{h(M)} \text{ and } g(N) = \overline{h(N)}$$

are ortho-adjacent, the intersection of $[g(M), g(M)^{\perp}]_k$ and $[g(N), g(N)^{\perp}]_k$ consists of one element and we have

$$f(S) = \overline{\overline{h(M)} \cup \overline{h(N)}} = \overline{h(S)}.$$

This means that f is induced by h. Therefore Theorem 3 can be proved by induction.

References

[1] Abramenko P. and Van Maldeghem H., *On oppositions in spherical buildings and twin buildings*, Ann. Combinatorics 4(2000), 125-137.

[2] Chow W. L., *On the geometry of algebraic homogeneous spaces*, Ann. of Math. 50 (1949) 32-67.

[3] Cohen A. M., *Point-Line Spaces Related to Buildings*, Handbook of incidence geometry. Buildings and foundations (Edited by F. Buekenhout), North-Holland, Amsterdam, 1995., p. 647-737.

[4] Faure C. A., Frölicher A., *Morphisms of Projective Geometries and Semilinear maps*, Geom. Dedicata, 53 (1994), 237-262.

[5] Faure C. A., *An Elementary Proof of the Fundamental Theorem of Projective Geometry*, Geom. Dedicata, 90 (2002), 145-151.

[6] Havlicek H., *A Generalization of Brauner's Theorem on Linear Mappings*, Mitt. Math. Sem. Univ. Gieβen, 215 (1994), 27-41.
[7] Huang W.-l., Kreuzer A., *Basis preserving maps of linear spaces*, Arch. Math. (Basel) 64(1995), 6, 530-533.
[8] Pankov M., Transformations of Grassmannians and automorphisms of classical groups, J. Geom. 75 (2002), 132-150.
[9] Pankov M., A characterization of geometrical mappings of Grassmann spaces, Results Math. 45(2004), 319–327.
[10] Pankov M., Mappings of the sets of invariant subspaces of null systems, Beiträge zur Algebra Geom., 45(2004), 389-399.
[11] Pankov M., Base preserving maps in symplectic geometry, submitted to Abh. Math. Sem. Univ. Hamburg.
[12] Pankov M., Prażmovski K., Żynel M., *Geometry of polar Grassmann spaces*, accepted to Demonstratio Math.
[13] Tits J., *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics 386, Springer, Berlin 1974.
[14] Van Maldeghem H., Private communication.

Department of Mathematics and Information Technology, University of Warmia and Mazury, Żolnierska 14A, 10-561 Olsztyn, Poland
E-mail address: pankov@matman.uwm.edu.pl